The General Solution of Bianchi Type III Vacuum Cosmology

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Abstract

The second order Ordinary Differential Equation which describes the unknown part of the solution space of some vacuum Bianchi Cosmologies is completely integrated for Type III, thus obtaining the general solution to Einstein's Field Equations for this case, with the aid of the sixth Painlevé transcendent P_{VI}. For particular representations of P_{VI} we obtain the known Kinnersley two-parameter space-time and a solution of Euclidean signature. The imposition of the space-time generalization of a "hidden" symmetry of the generic Type III spatial slice, enables us to retrieve the two-parameter subfamily without considering the Painlevé transcendent.

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1 Introduction

In a recent work of ours [1], the theory of symmetries of systems of coupled, ordinary differential equations (ODE's) has been used to develop a concise algorithm for cartographing the space of solutions to vacuum Bianchi Einstein's Field Equations (EFE). The symmetries used were the well known automorphisms of the Lie algebra for the corresponding isometry group of each Bianchi Type, as well as the scaling and the time reparameterization symmetry. Application of the method to Type III resulted in: a) the recovery of most known solutions without prior assumption of any extra symmetry, b) the enclosure of the entire unknown part of the solution space into a single, second order ODE in terms of one dependent variable and c) a partial solution to this ODE. It is worth-mentioning the fact that the solution space were thus seen to be naturally partitioned into three distinct, disconnected pieces: one consisting of the known Siklos (pp-wave) solution [2], another occupied by the Type III member of the known Ellis-MacCallum family [3],[4] and the third described by the aforementioned ODE. Lastly, preliminary results reported have shown that the unknown part of the solution space for other Bianchi Types is described by a strikingly similar ODE, pointing to a natural operational unification, at least as far as the problem of solving the cosmological EFE's is concerned. In section 2 of this work we present the general solution to the ODE in question for the case of Type III, that is the general Type III vacuum Geometry. The metric components are implicitly given in terms of the sixth Painlevé transcendent. To overcome this practical difficulty we use the existing knowledge on particular, truly closed form (i.e. in terms of elementary functions of time) representations of the P_{VI} transcendent; work on finding such solutions can be found in the contributions to a recent meeting at the Newton Institute [5], and also in [6], [7]. For other cases of Painlevé solutions in relativity see, e.g. [8], [9], [10] and (13.60), (13.70) in [3]. For a recent account on how Painlevé transcendents occur in dimensional reductions of integrable systems see [11]. This investigation results in the recovery of the known Kinnersley [12] two parameter family of metrics, admitting a G_4 multiply transitive isometry group and a bi-parametric solution of Euclidean signature, admitting only the initial G_3 isometry. In section 3, the existence of an extra inherent symmetry of the general Type III 3-geometry is exploited in order to arrive at the aforementioned Kinnersley solution by a prior assumption of extra symmetry. Finally, some concluding remarks are included in section 4, while some mathematical aspects of the derivation in section 3 are described in the Appendix.

2 The General Solution

As it is well known, for spatially homogeneous space-times with a simply transitive action of the corresponding isometry group [3], [13], the line element assumes the form

$$ds^2 = \left(N^\alpha N_\alpha - N^2\right) dt^2 + 2\, N_\alpha\, \sigma^\alpha{}_i\, dx^i\, dt + \gamma_{\alpha\beta}\, \sigma^\alpha{}_i\, \sigma^\beta{}_j\, dx^i\, dx^j \tag{2.1}$$

where the 1-forms $\sigma^{\alpha}{}_{i}$, are defined from:

$$d\sigma^{\alpha} = C^{\alpha}{}_{\beta\gamma}\,\sigma^{\beta}\wedge\sigma^{\gamma} \Leftrightarrow \sigma^{\alpha}{}_{i,j} - \sigma^{\alpha}{}_{j,i} = 2\,C^{\alpha}{}_{\beta\gamma}\,\sigma^{\gamma}{}_{i}\sigma^{\beta}{}_{j}. \tag{2.2}$$

(small Latin letters denote world space indices while small Greek letters count the different basis one-forms; both type of indices range over the values 1,2,3)

For Bianhi Type III Cosmology the structures constants are [14]

$$\begin{aligned} &C^{1}{}_{13} = -C^{1}{}_{31} = 1 \\ &C^{\alpha}{}_{\beta\gamma} = 0 \qquad\qquad for\; all\; other\; values\; of\; \alpha\beta\gamma \end{aligned} \tag{2.3}$$

Using these values in the defining relation (2.2) of the 1-forms $\sigma^{\alpha}{}_{i}$ we obtain

$$\sigma^{\alpha}{}_{i} = \begin{pmatrix} 0 & e^{-x} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \tag{2.4}$$

The corresponding vector fields $\xi^{i}{}_{\alpha}$ (satisfying $[\xi_{\alpha}, \xi_{\beta}] = C^{\gamma}{}_{\alpha\beta}\,\xi_{\gamma}$) with respect to which the Lie Derivative of the above 1-forms is zero are:

$$\xi_1 = \partial_y \quad \xi_2 = \partial_z \quad \xi_3 = \partial_x + y\partial_y \tag{2.5}$$

In the recent work of ours [1], we retrieved most known solutions of Bianchi Type III and we showed that the unknown part of the solution space of this Cosmology is described, without loss of generality, by a line element of the form:

$$ds^2 = -N^2\,d\rho^2 + \gamma_{\alpha\beta}\,\sigma^{\alpha}{}_{i}\,\sigma^{\beta}{}_{j}\,dx^i\,dx^j \tag{2.6}$$

where the scale factor matrix $\gamma_{\alpha\beta}(\rho)$ and the lapse function $N(\rho)$ are given by the equations:

$$(N)^2 = \frac{u'^2 - 1}{8\,(3\,\rho - 5)\,(\rho^2 - u^2 - 1)}\,e^{u_1} \quad \text{and} \quad \gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1 + 2\,u_4} & e^{u_1+u_2+u_4} & 0 \\ e^{u_1+u_2+u_4} & \frac{3\,\rho-3}{3\,\rho-5}\,e^{u_1+2\,u_2} & 0 \\ 0 & 0 & e^{u_1} \end{pmatrix} \tag{2.7}$$

The functions u_1, u_2, u_4 satisfy

$$u_1' = \frac{-3\,u + (3\,\rho - 1)\;u'}{2\,\left(u'^2 - 1\right)}\,u'' \tag{2.8}$$

$$u_2' = \frac{(3\,\rho - 1)\;\left(-1 + u'^2 + (3\,\rho - 5)^2\,u\,u'' - (3\,\rho - 5)^2\;(\rho - 1)\;u'\,u''\right)}{4\,(3\,\rho - 5)^2\;(\rho - 1)\;\left(u'^2 - 1\right)} \tag{2.9}$$

$$u_4' = \frac{3\,\rho - 5}{3\,\rho - 1}\,u_2' \tag{2.10}$$

and the function $u(\rho)$ obeys a second order differential equation, of the form:

$$\ddot{u}^2 = \frac{(-1+\dot{u}^2)^2}{(\kappa+\lambda\,\rho)\,(\rho^2-u^2-1)} \quad \kappa=-10,\ \lambda=6 \tag{2.11}$$

In order to solve (2.11), for arbitrary constants $(\kappa,\,\lambda)$, we apply the contact transformation:

$$\begin{aligned}
u(\rho) &= -\frac{8}{\lambda}\,y(x) + \frac{4\,(2x-1)}{\lambda}\,y'(x) \qquad & \rho &= -\frac{\kappa}{\lambda} + \frac{4}{\lambda}\,y'(x) \\
u'(\rho) &= 2\,x-1 & u''(\rho) &= \frac{\lambda}{2\,y''(x)}
\end{aligned} \tag{2.12}$$

which reduces it to

$$x^2\,(x-1)^2\,y''^2 = -4y'\,(x\,y'-y)^2 + 4\,y'^2\,(x\,y'-y) - \frac{\kappa}{2}\,y'^2 + \frac{\kappa^2-\lambda^2}{16}\,y' \tag{2.13}$$

This equation is a special form of the equation SD-Ia appearing in [15], a work in which a classification of second order, second degree ordinary differential equations has been performed. The general solution to (2.13) is obtained with the help of the sixth Painlevé transcendent $P := \mathbf{P_{VI}}(\alpha,\beta,\gamma,\delta)$ and reads:

$$\begin{aligned}
y \;=\;& \frac{x^2\,(x-1)^2}{4\,P\,(P-1)(P-x)}\left(P' - \frac{P\,(P-1)}{x\,(x-1)}\right)^2 \\
&+\frac{1}{8}\,(1\pm\sqrt{2\,\alpha})^2\,(1-2\,P) - \frac{\beta}{4}\left(1-\frac{2\,x}{P}\right) \\
&-\frac{\gamma}{4}\left(1-\frac{2\,(x-1)}{P-1}\right) + \left(\frac{1}{8}-\frac{\delta}{4}\right)\left(1-\frac{2\,x\,(P-1)}{P-x}\right)
\end{aligned} \tag{2.14}$$

$P := \mathbf{P_{VI}}(\alpha,\beta,\gamma,\delta)$ is defined by the ODE:

$$\begin{aligned}
P'' \;=\;& \frac{1}{2}\left(\frac{1}{-1+P}+\frac{1}{P}+\frac{1}{-x+P}\right)P'^2 - \left(\frac{1}{-1+x}+\frac{1}{x}+\frac{1}{-x+P}\right)P' \\
&+\frac{(-1+P)\;P\;(-x+P)}{(-1+x)^2\,x^2}\left(\alpha + \frac{(-1+x)\;\gamma}{(-1+P)^2} + \frac{x\,\beta}{P^2} + \frac{(-1+x)\;x\,\delta}{(-x+P)^2}\right)
\end{aligned} \tag{2.15}$$

where the values of the parameters $(\alpha,\beta,\gamma,\delta)$ must satisfy the following system:

$$\alpha-\beta+\gamma-\delta\pm\sqrt{2\,\alpha}+1 \;=\; -\frac{\kappa}{2} \tag{2.16a}$$

$$(\beta+\gamma)\left(\alpha+\delta\pm\sqrt{2\,\alpha}\right) \;=\; 0 \tag{2.16b}$$

$$(\gamma-\beta)\left(\alpha-\delta\pm\sqrt{2\,\alpha}+1\right) + \frac{1}{4}\left(\alpha-\beta-\gamma+\delta\pm\sqrt{2\,\alpha}\right)^2 \;=\; \frac{\kappa^2-\lambda^2}{16} \tag{2.16c}$$

$$\frac{1}{4}\,(\gamma-\beta)\left(\alpha+\delta\pm\sqrt{2\,\alpha}\right)^2 + \frac{1}{4}\,(\beta+\gamma)^2\left(\alpha-\delta\pm\sqrt{2\,\alpha}+1\right) \;=\; 0 \tag{2.16d}$$

If we insert in (2.16) the values $\kappa = -10, \lambda = 6$ for Type III, we have twelve solutions to this system. Eight of them correspond to the "$-\sqrt{2\,\alpha}$" case and the rest four to the "$+\sqrt{2\,\alpha}$" case.

Case I: $-\sqrt{2\,\alpha}$

$$\alpha = 0,\, \beta = 0,\, \gamma = 0,\, \delta = -4 \tag{2.17a}$$
$$\alpha = 0,\, \beta = -2,\, \gamma = 2,\, \delta = 0 \tag{2.17b}$$
$$\alpha = 2,\, \beta = 0,\, \gamma = 0,\, \delta = -4 \tag{2.17c}$$
$$\alpha = 2,\, \beta = -2,\, \gamma = 2,\, \delta = 0 \tag{2.17d}$$
$$\alpha = 8,\, \beta = 0,\, \gamma = 0,\, \delta = 0 \tag{2.17e}$$
$$\alpha = \frac{1}{2},\, \beta = -\frac{9}{2},\, \gamma = \frac{1}{2},\, \delta = \frac{1}{2} \tag{2.17f}$$
$$\alpha = \frac{1}{2},\, \beta = -\frac{1}{2},\, \gamma = \frac{9}{2},\, \delta = \frac{1}{2} \tag{2.17g}$$
$$\alpha = \frac{9}{2},\, \beta = -\frac{1}{2},\, \gamma = \frac{1}{2},\, \delta = -\frac{3}{2} \tag{2.17h}$$

Case II: $+\sqrt{2\,\alpha}$

$$\alpha = 0,\, \beta = 0,\, \gamma = 0,\, \delta = -4 \tag{2.18a}$$
$$\alpha = 0,\, \beta = -2,\, \gamma = 2,\, \delta = 0 \tag{2.18b}$$
$$\alpha = 2,\, \beta = 0,\, \gamma = 0,\, \delta = 0 \tag{2.18c}$$
$$\alpha = \frac{1}{2},\, \beta = -\frac{1}{2},\, \gamma = \frac{1}{2},\, \delta = -\frac{3}{2} \tag{2.18d}$$

Of course the first two solutions in (2.18) are identical to the first two in (2.17) and are only written down for the sake of completeness.

Despite the fact that (2.14) describes the general solution of Bianchi Type III Vacuum Cosmology, this solution does not come in a particularly manageable form due to the appearance of the function $\mathbf{P_{VI}}(\alpha, \beta, \gamma, \delta)$. As mentioned in the introduction, there is a vast body of literature concerning $\mathbf{P_{VI}}$. The most effective way to find its closed form solutions, is to apply the following lemma (often called "linearisability condition") [16], [17]:

Lemma: Assume that $P(x)$ satisfies the Riccati equation

$$x\,(x-1)\,P'(x) = a\,P(x)^2 + (b\,x + c)\,P(x) - (a+b+c)\,x \tag{2.19}$$

Then $P(x)$ satisfies:

$$\boldsymbol{P_{VI}}\left(\frac{a^2}{2}, -\frac{(a+b+c)^2}{2}, \frac{(a+c)^2}{2}, \frac{1-(1-a-b)^2}{2}\right) \tag{2.20}$$

The proof is elementary: Solving (2.19) and its first derivative for $P'(x), P''(x)$ and substituting in (2.15) we get an identity for the specific values of the parameters $(\alpha, \beta, \gamma, \delta)$ satisfying the values of (2.20).

The application of this lemma proceeds as follows: For every case of (2.17) and (2.18), we find the values of (a, b, c) (if they exist). Then we solve the corresponding (2.19). Subsequently, we insert each solution to (2.14), thereby obtaining $y(x)$. Finally inserting this $y(x)$ into (2.12) we obtain $(u(\rho), \rho)$ in parametric form in terms of the time variable x. The results of this procedure are:

Case I: $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = -4$

It cannot be linearized.

Case II: $\alpha = 0$, $\beta = -2$, $\gamma = 2$, $\delta = 0$

We get two solutions of (2.15)

$$P(x) = \frac{x\,(2 + 2\,x\,\log(x) + c\,x)}{(-1+x)^2} \tag{2.21a}$$

$$P(x) = \frac{-3 + 4\,x - 2\,(-1+x)^2\,\log(-1+x) + c\,(-1+x)^2}{x^2} \tag{2.21b}$$

but both of them make the Bäcklund transformation (2.12) degenerate, since they give $\rho \to \frac{5}{3}$.

Case III: $\alpha = 2$, $\beta = 0$, $\gamma = 0$, $\delta = -4$

We get one solution

$$P(x) = \frac{(-1+x)^2}{1 - 2\,x + c\,x^2} \tag{2.22}$$

which makes (2.12)

$$u(\rho) = \frac{4\,\left(-1 - 2\,(-1+c)\,x + 6\,c\,x^2 - 4\,c\,x^3 + c\,(2+c)\,x^4\right)}{3\,(-1 + 2\,x + c\,x^2)^2} \tag{2.23a}$$

$$\rho = \frac{5 + 4\,(-3+2\,c)\,x - 6\,(-2+3\,c)\,x^2 + 20\,c\,x^3 + 5\,c^2\,x^4}{3\,(-1 + 2\,x + c\,x^2)^2} \tag{2.23b}$$

Case IV: $\alpha = 2$, $\beta = -2$, $\gamma = 2$, $\delta = 0$

We get three solutions of (2.15). The first two

$$P(x) = \frac{x\,\left(-6 + c\,(-2+x) - 5\,x + 4\,(-2+x)\,\log(2-2\,x)\right)}{-9 - 6\,x + 2\,x^2 + c\,(-3+2\,x) + 4\,(-3+2\,x)\,\log(2-2\,x)} \tag{2.24a}$$

$$P(x) = \frac{-4 + (-30+4\,c)\,x + (13+2\,c)\,x^2 - 8\,x\,(2+x)\,\log(x)}{-27 + 2\,x + 4\,x^2 + c\,(2+4\,x) - 8\,(1+2\,x)\,\log(x)} \tag{2.24b}$$

are unacceptable, since they give again $\rho \to \frac{5}{3}$, and the third:

$$P(x) = \frac{1 - 2\,(3 + 4\,c)\,x + 12\,c\,x^2 + 2\,x\,(-2 + 3\,x)\,\log\left(\frac{x}{1-x}\right)}{-4 + c\,(-4 + 8\,x) + (-2 + 4\,x)\,\log\left(\frac{x}{1-x}\right)} \tag{2.25}$$

which makes (2.12)

$$u(\rho) = \frac{A(x)}{B(x)}, \qquad \rho = \frac{C(x)}{D(x)} \tag{2.26}$$

with

$$\begin{aligned} A(x) &= 4\,x\,\left(1 - 3\,x + 2\,x^2\right)\,\log^2\left(\frac{x}{1-x}\right) \\ &\quad + 4\,(-1 + x)\,x\,(-1 + c\,(-4 + 8\,x))\,\log\left(\frac{x}{1-x}\right) \\ &\quad + 4\,\left(-1 + 2\,c + 4\,c^2\right)\,x - 8\,c\,(1 + 6\,c)\,x^2 + 32\,c^2\,x^3 + 2 \end{aligned} \tag{2.27a}$$

$$\begin{aligned} B(x) &= 3\,(2 + 2\,c - 4\,c\,x + (-1 + 2\,x)\,\log(1 - x) + \log(x) - 2\,x\,\log(x))^2 \\ &\quad x\,(-1 + x) \end{aligned} \tag{2.27b}$$

$$\begin{aligned} C(x) &= x\,\left(-5 + 17\,x - 24\,x^2 + 12\,x^3\right)\log^2\frac{x}{1-x} \\ &\quad + 4\,(-1 + x)\,x\,\left(3 - 6\,x + c\,\left(5 - 12\,x + 12\,x^2\right)\right)\log\frac{x}{1-x} \\ &\quad + 2\,\left(-1 - 2\,\left(3 + 6\,c + 5\,c^2\right)\,x + \left(6 + 36\,c + 34\,c^2\right)\,x^2 - 24\,c\,(1 + 2\,c)\,x^3 + 24\,c^2\,x^4\right) \end{aligned} \tag{2.27c}$$

$$\begin{aligned} D(x) &= 3\,(2 + 2\,c - 4\,c\,x + (-1 + 2\,x)\,\log(1 - x) + \log(x) - 2\,x\,\log(x))^2 \\ &\quad x\,(-1 + x) \end{aligned} \tag{2.27d}$$

Case V: $\alpha = 8$, $\beta = 0$, $\gamma = 0$, $\delta = 0$

It cannot be linearized.

Case VI: $\alpha = \frac{1}{2}$, $\beta = -\frac{9}{2}$, $\gamma = \frac{1}{2}$, $\delta = \frac{1}{2}$

We get one solution

$$P(x) = \frac{-2 + 3\,x + 2\,c\,x^3}{-1 + 2\,x + 2\,c\,x^2} \tag{2.28}$$

which gives (2.23) again.

Case VII: $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{9}{2}$, $\delta = \frac{1}{2}$

We get one solution

$$P(x) = \frac{x\,(-1 - 4\,c\,x + 2\,c\,x^2)}{-1 + 2\,x + 2\,c\,x^2} \tag{2.29}$$

which gives (2.23) again.

Case VIII: $\alpha = \frac{9}{2}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2}$, $\delta = -\frac{3}{2}$

We get three solutions of (2.15).

One of which is identical to Case VI, and two more

$$P(x) = \frac{x\,\left(5 + 2\,c\,(-1+x)^2 - 4\,x - 4\,x^2 + 6\,(-1+x)^2\,\log(-1+x)\right)}{3\,\left(5 + 2\,c\,(-1+x)^2 - 4\,x - 4\,x^2 + 2\,x^3 + 6\,(-1+x)^2\,\log(-1+x)\right)} \tag{2.30a}$$

$$P(x) = \frac{x\,(-3 + 2\,(-9+2\,c)\,x + 2\,(6+c)\,x^2 - 6\,x\,(2+x)\,\log(x))}{3\,(1 - 6\,x + 2\,c\,x^2 + 2\,x^3 - 6\,x^2\,\log(x))} \tag{2.30b}$$

which are again unacceptable, since they give $\rho \to \frac{5}{3}$.

Case IX: $\alpha = 2$, $\beta = 0$, $\gamma = 0$, $\delta = 0$

We get one solution

$$P(x) = \frac{(1+2\,c)\,x^2}{-1 + 2\,x + 2\,c\,x^2} \tag{2.31}$$

which again leads to (2.23).

Case X: $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2}$, $\delta = -\frac{3}{2}$

We get four solutions of (2.15). The first

$$P(x) = \frac{x + 4\,c\,x^2 - 2\,c\,x^3}{-1 + 2\,x + 2\,c\,x^2} \tag{2.32}$$

gives (2.23), the second

$$P(x) = \frac{3 - 6\,(1+2\,c)\,x + 12\,c\,x^2 + 6\,(-1+x)\,x\,\log\left(\frac{x}{1-x}\right)}{-4 + c\,(-4+8\,x) + (-2+4\,x)\,\log\left(\frac{x}{1-x}\right)} \tag{2.33}$$

gives (2.26) and two more

$$P(x) = \frac{1 - (2+c)\,x + x\,\log(x)}{c + x - \log(x)} \tag{2.34}$$

$$P(x) = -\frac{c\,(-2+x) - 2\,x + (-2+x)\,\log(-1+x)}{c + x + \log(-1+x)} \tag{2.35}$$

which are degenerate, since they give $\rho \to \frac{5}{3}$.

The conclusion of the above analysis is that we have two particular solutions for the function $u(\rho)$, i.e. (2.23) and (2.26).

In order to write down the line element that corresponds to (2.23) we change the parameter x and the constant c to the values

$$x \to \frac{1 - cosh(\xi)}{2} \quad c \to \frac{-1}{\lambda + 1} \tag{2.36}$$

Gathering all the pieces we arrive at the final form of the metric:

$$ds^2 = \kappa^2 \left(A(\xi)\,(-d\xi^2 + dx^2) + B(\xi)\,e^{-2\,x}\,dy^2 + 2\,C(\xi)\,e^{-x}\,dy\,dz + C(\xi)\,dz^2 \right) \tag{2.37}$$

where

$$\begin{aligned}
A(\xi) &= \frac{1}{4}\,(\cosh 2\,\xi + 4\,\lambda\,\cosh\xi + 3) \\
B(\xi) &= \frac{\cosh 4\,\xi + 8\,\lambda\,\cosh 3\,\xi + 28\,\cosh 2\,\xi + 56\,\lambda\,\cosh\xi + 32\,\lambda^2 + 3}{2\,(\cosh 2\,\xi + 4\,\lambda\,\cosh\xi + 3)} \\
C(\xi) &= \frac{16\,(1 - \lambda^2)\,\sinh^2\xi}{\cosh 2\,\xi + 4\,\lambda\,\cosh\xi + 3}
\end{aligned} \tag{2.38}$$

with $-1 < \lambda < 1$ for the metric to have signature $(- + + +)$.

The metric (2.37) admits a fourth killing vector

$$\xi_4 = -y\,\partial_x + \left(\frac{1}{8}\,e^{2\,x} - \frac{1}{2}\,y^2 \right)\,\partial_y - \frac{1}{4}\,e^x\partial_z \tag{2.39}$$

which produces with (2.5) the following table of (non-vanishing) commutators:

$$[\xi_1, \xi_3] = \xi_1,\ [\xi_1, \xi_4] = -\xi_3,\ [\xi_3, \xi_4] = \xi_4 \tag{2.40}$$

The existence of a fourth Killing field implies that this geometry is an LRS space time (see [13] where all LRS Bianchi geometries are characterized)

The isotropy group inferred from the above algebra (see the last commutator) is a G_1 spatial rotation.

The line element (2.37) is a two-parametric family and is one of the Kinnersley vacuum solutions [12]. One way we can be assured that the constants κ, λ are indeed essential, is to consider the following three scalars, constructed out of the Riemmann tensor and its first and second covariant derivatives:

$$Q_1 = R^{ABCD}\,R_{ABCD}, \qquad Q_2 = {Q_{1\,;\,A}}^{;\,A}, \qquad Q_3 = Q_{1\,;A}\,{Q_1}^{;\,A} \tag{2.41}$$

(here capital Latin letters stand for world space time indices, ranging over the values 0,1,2,3)

The determinant of the Wronskian matrix $\frac{\partial(Q_1,Q_2,Q_3)}{\partial(\xi,\kappa,\lambda)}$ can been seen to be non zero. Therefore, (κ,λ) (and of course ξ) can be in principle expressed as functions of (2.41), and are thus essential.

The line element that corresponds to solution (2.26), is quite interesting because as it can be seen leads to an imaginary lapse function, implying a Euclidean signature for the metric tensor. To get a manageable form of this metric we change the parameter x and the constant c to the values

$$x \to \arctan e^{-\mu^2\,\cos^2\xi} \quad c \to -\mu^2 \tag{2.42}$$

thus arriving at the line element

$$ds^2 = \kappa^2\left(A(\xi)\,d\xi^2 + B(\xi)\,dx^2 + \frac{e^{-2\,x}}{B(\xi)}\,dy^2 + 2\,C(\xi)\,e^{-x}\,dy\,dz + D(\xi)\,dz^2\right) \tag{2.43}$$

where

$$\begin{aligned}
A(\xi) &= \frac{1}{2}\,\mu^4\,\mathrm{sech}^2(\mu^2\,\cos^2\xi)\,\sin^2 2\,\xi\,\left(1+\mu^2\,\sin^2\xi\,\tanh(\mu^2\,\cos^2\xi)\right)\\
B(\xi) &= \frac{1}{2}\,\left(1+\mu^2\,\sin^2\xi\,\tanh(\mu^2\,\cos^2\xi)\right)\\
C(\xi) &= \frac{\mathrm{sech}^2(\mu^2\,\cos^2\xi)\,\left(2\,\mu^2\,\sin^2\xi-\sinh(2\,\mu^2\cos^2\xi)\right)}{1+\mu^2\,\sin^2\xi\,\tanh(\mu^2\,\cos^2\xi)}\\
D(\xi) &= \frac{\mathrm{sech}^2(\mu^2\,\cos^2\xi)\,\left(1+\cosh(2\,\mu^2\cos^2\xi)+2\,\mu^4\,\sin^4\xi\right)}{1+\mu^2\,\sin^2\xi\,\tanh(\mu^2\,\cos^2\xi)}
\end{aligned} \tag{2.44}$$

with $(\mu,\xi)\in\mathbb{R}$.

The essential nature of the constants κ,μ, is secured by the fact that the determinant of the Wronskian matrix $\frac{\partial(Q_1,Q_2,Q_3)}{\partial(\xi,\kappa,\mu)}$ for the corresponding curvature scalars (2.41) is again nonzero. Two other interesting feature of this metric are: (a) the fact that it *does not* admit any other killing fields besides the initially assumed (2.5), i.e. is a pure G_3 geometry and (b) that it is *not* either *self-dual* or *anti self-dual*.

At this stage we have extracted from solution (2.13) as much information as we possibly could, regarding its particular solutions. The conclusion is that the only line element, in terms of elementary functions, with Lorentzian signature is (2.37) which admits a G_4 symmetry. This solution, as already mentioned, was first found in [12] during the search of all Petrov Type D metrics which by no doubt is a completely different approach from ours. On the other hand the way we reproduced this solution is by searching for particular solutions of the Painlevé equation, a purely mathematical way of viewing, in which no prior assumption of symmetry has been adopted. We find it interesting to retrieve the same line element by previously assuming the existence of a fourth killing field. This is what we will do in the next section, exploiting a "hidden" symmetry of the Bianchi Type III 3-Geometry.

3 Derivation of the Lorentzian solution assuming a G_4 symmetry

In [1] we were not able to obtain the full solution (2.11), thus we presented only an one-parameter family of Bianchi Type III Cosmology. The line element is of the general form (2.6) with an overall essential constant κ^2:

$$
\begin{aligned}
ds^2 \;&=\; \kappa^2 \left(-\frac{e^{2\,\xi}\,(e^{2\,\xi}+1)}{4\,(2\,e^{2\,\xi}+1)}\,d\xi^2 + \frac{e^{\,\xi}}{4}\,cosh\xi\,dx^2 + e^{-2x+\xi}(cosh2\xi+2)\,sech\xi\,dy^2 \right. \\
&\left. \qquad + \; e^{\xi}\,sech\xi\,dz^2 + 2e^{-x+\xi}\,sech\xi\,dy\,dz \right)
\end{aligned} \tag{3.1}
$$

This metric admits, besides the three killing fields (2.5), yet another one, namely:

$$
\xi_4 = -y\,\partial_x + \frac{e^{2\,x}-8\,y^2}{16}\,\partial_y - \frac{1}{8}\,e^x\,\partial_z \tag{3.2}
$$

The commutator table of their algebra is:

$$
\begin{array}{lll}
\big[\xi_1,\xi_2\big]=0 & \big[\xi_1,\xi_3\big]=\xi_1 & \big[\xi_1,\xi_4\big]=-\xi_3 \\
\big[\xi_2,\xi_3\big]=0 & \big[\xi_2,\xi_4\big]=0 & \big[\xi_3,\xi_4\big]=\xi_4
\end{array} \tag{3.3}
$$

An equivalent form of the Type III member of the known Ellis-MacCallum family of solutions [3],[4], also retrieved in [1], reads:

$$
ds^2 = \lambda^2 \left(-\frac{e^{3\,t}}{e^t-1}\,dt^2 + e^{2\,t}\,dx^2 + e^{2\,t-2x}\,dy^2 + (1-e^{-\,t})\,dz^2 \right) \tag{3.4}
$$

which again admits a fourth killing field:

$$
\eta = -y\,\partial_x + \frac{e^{2\,x}-y^2}{2}\,\partial_y \tag{3.5}
$$

The interesting thing is that both quadruplets $(\xi_1,\xi_2,\xi_3,\xi_4)$, (ξ_1,ξ_2,ξ_3,η) span the same algebra. Yet the two line elements (3.1) and (3.4) are inequivalent, i.e. we cannot arrive from one to the other by a coordinate transformation. This can be easily seen since for metric (3.4) we have an invariant relation of the form:

$$
\frac{18\,Q_1^7}{(Q_2\,Q_1 - Q_1{}^{;\,A}\,Q_{1;\,A})^3} = \lambda^2,\; Q_1 = R^{KLMN}R_{KLMN},\; Q_2 = \Box R^{KLMN}R_{KLMN} \tag{3.6}
$$

where capital Latin letters denote space-time indices ranging in the interval (0-3), the semicolon stands for covariant differentiation, and the $\Box$ for the covariant D'Alebertian. This relation, being a constant scalar constructed out of the intrinsic geometry (the

Riemmann tensor and its covariant derivatives), characterizes, along with many others that can be found, this metric: It will be valid for any equivalent, under general coordinate transformations, form of (3.4). But for metric (3.1) the left hand side of (3.6) does not equal κ^2, so the two metrics are inequivalent.

The way that the two solutions were found, belonging in different "branches" of the solution space, and without any assumption of extra symmetry, suggests that the existence of the fourth killing field may not be a mere coincidence: Indeed the very existence of a 3-space with a Type III symmetry group implies the existence of a G_4 action, a thing that is not so common.

In order to see this we consider an arbitrary hypersurface with $t = t_o$ of the space-time (2.6):

$$dl^2 = \gamma_{\alpha\beta}\, \sigma^{\alpha}{}_{i}\, \sigma^{\beta}{}_{j}\, dx^i\, dx^j \tag{3.7}$$

where the scale factor matrix is given by

$$\gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1+2\,u_4} & e^{u_1+u_2+u_4} & 0 \\ e^{u_1+u_2+u_4} & e^{u_1+2\,u_2}\, t_o & 0 \\ 0 & 0 & e^{u_1} \end{pmatrix} \tag{3.8}$$

and the functions u_i are evaluated at $t = t_o$. Solving the killing equation for this line element, we surprisingly find that, besides the three killing fields (2.5), there exists a fourth:

$$\zeta = 2\,y\,\partial_x + \left(y^2 - \frac{e^{-2\,u_4}\,t_o}{4\,(t_o-1)}\,e^{2\,x}\right)\partial_y + \frac{e^{-u_2-u_4}}{2\,(t_o-1)}\,e^{x}\,\partial_z \tag{3.9}$$

This field would exist even if we have filled out the zero entries of the scale factor matrix $\gamma_{\alpha\beta}$ using the constant Automorphisms. The qualitative difference between (3.9) and (2.5) is that the components of the former depend on $\gamma_{\alpha\beta}$, and thus this vector is not form invariant like the latter.

If we wish this 3-dimensional killing field, to be promoted to an isometry of space-time, we first need to prolong it by adding a time component say of the form $f(t,x,y,z)\,\frac{\partial}{\partial t}$. Omitting the calculational details, the result is that $f(t,x,y,z)$ should be zero (trivial prolongation) and the components of ζ must be independent of the slice parametrization. Thus the derivative with respect to t_o, for every component of ζ, is zero, i.e.

$$\left\{\begin{array}{rcl} \dfrac{d}{dt_o}\left(\dfrac{e^{-2\,u_4(t_o)}}{4\,(t_o-1)}\,t_o\right) & = & 0 \\ \dfrac{d}{dt_o}\left(\dfrac{e^{-u_2(t_o)-\,u_4(t_o)}}{2\,(t_o-1)}\right) & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} u_2(t) & = & k_2 - \frac{1}{2}\,\ln(t^2-t) \\ u_4(t) & = & k_4 - \frac{1}{2}\,\ln\left(\frac{t-1}{t}\right) \end{array}\right. \tag{3.10}$$

Inserting these values into the scale factor matrix $\gamma_{\alpha\beta}$ we have:

$$\gamma_{\alpha\beta} = \begin{pmatrix} \frac{t}{t-1}\,e^{u_1} & \frac{1}{t-1}\,e^{u_1} & 0 \\ \frac{1}{t-1}\,e^{u_1} & \frac{1}{t-1}\,e^{u_1} & 0 \\ 0 & 0 & e^{u_1} \end{pmatrix} \tag{3.11}$$

where we have absorbed the constants (k_2, k_4) with a re-scaling of the coordinates $(y, z) \to (e^{-k_4}\, y, e^{-k_2}\, z)$. It is easy to see that this scale factor matrix is positive definite in the interval $t \in (1, +\infty)$. Using (3.11) in the line element (2.6) we are now ready to solve the vacuum Einstein Field Equations (EFE) $R_{AB} = 0$, $\{A, B\} \in \{0, 1, 2, 3\}$.

From the $R_{00} = 0$ component we deduce:

$$N^2 = \frac{(3\,(t-1)\,u_1' - 2)\ u_1'}{4\,(4\,t-3)}\, e^{u_1} \tag{3.12}$$

Inserting this value of the lapse function, into the rest of EFE, we arrive at a *single* second order ODE for the function $u_1(t)$:

$$u_1'' = -\frac{6\,t\,(t-1)^2\,{u_1'}^2 + (10\,t^2 + 13\,t - 3)\,u_1' + 8\,t - 5}{4\,t^2 - 7\,t + 3}\, u_1' \tag{3.13}$$

which is an Abel equation for the function $w(t) := u_1'(t)$. The general solution to this equation (see Appendix) for the function $w(t)$ is given in implicit form:

$$G(t, w) := \frac{(3\,(t-1)\,w - 2)^3\ (t-1)^3\,w}{\left(2\,(t-1)^2\,(t-3)\,w^2 - 2\,(t^2 - 4\,t + 3)\,w - 1\right)^2\ (4\,t - 3)} = \mathit{constant} \tag{3.14}$$

One possible parametrization of $G(t, w) = \mathit{const}$ is

$$\begin{aligned} w &= \frac{64\,(1-\lambda^2)\,(\lambda + \cosh\xi)\,\sinh^4\xi}{(\cosh 3\,\xi - 9\,\cosh\xi - 8\,\lambda)\,(\cosh 2\,\xi + 4\,\lambda\,\cosh\xi + 3)^2} \\ t &= \frac{\cosh 4\,\xi + 8\,\lambda\,(\cosh 3\,\xi + 7\,\cosh\xi) + 28\,\cosh 2\,\xi + 32\,\lambda^2 + 3}{32\,(1-\lambda^2)\,\sinh^2\xi} \end{aligned} \tag{3.15}$$

yielding

$$G(t, w) = \frac{1-\lambda^2}{12\,\lambda^2} \tag{3.16}$$

Of course, as it is evident from (3.15), this parametrization is not valid for $\lambda = \pm 1$, but then, from (3.16) we deduce that $G = 0 \Rightarrow u_1' = \frac{2}{3\,(t-1)}$ which is unacceptable since it makes the lapse function (3.12) zero. Another breakdown of (3.15), is when the denominator of $G(t, w)$ vanishes. However, this is the case that corresponds to the line element (3.1), i.e. emerges from a special solution of (3.13). Gathering all the pieces we arrive at the line element (2.37).

4 Conclusions

We have seen how the Automorphisms of Type III Geometry can be used as symmetries of the corresponding EFE's, in order to reduce the degree of these equations, and ultimately integrate them in full. The solution space of the differential equations, is seen to be naturally partitioned in three disconnected components: One occupied by the Type III member of the known Ellis-MacCallum family (3.4), another described by the equation (2.11) which is fully integrated by the parametrization (2.14) and a piece occupied by the known Siklos solution, an equivalent form of which is

$$ds^2 = -\lambda^2\, d\xi^2 + \frac{\xi^2}{4}\, dx^2 + e^{-2x}\xi^{4\lambda}\, dy^2 + \frac{\lambda - 1}{2\lambda - 1}\, dz^2 + 2e^{-x}\xi^{2\lambda}\, dy\, dz \tag{4.1}$$

This line element can be obtained from (2.7), for the special case in which $u_3 := \frac{\gamma_{11}\,\gamma_{22}}{\gamma_{12}^2} = cont$; u_3 in this paper is, by a choice of time "gauge", taken to be the term $\frac{3\,\rho-3}{3\,\rho-5}$ in $\gamma_{\alpha\beta}$.

But at the level of the geometry, a unification of the first two branches might be achieved, at the expense of complete mathematical rigor. Consider the particular solution to the Painlevé equation (2.15) which corresponds to the two-parameter family (2.37) (Kinnersley solution): This solution can be seen to incorporate both (3.1) found in [1], for an admissible choice of λ and the Type III member of Ellis-MacCallum metric (3.4) for the marginal value $\lambda = 1$ (since this is the only value for which the invariant relation (3.6) is satisfied by the two-parameter family). However, for $\lambda = 1$ the coefficient of dz^2 is vanishing along with the cross term $dy\, dz$. In order to avoid this incompatibility (zero eigenvalues) one may first employ the transformation

$$z \to \frac{z}{2\sqrt{1-\lambda^2}}, \quad y \to \tfrac{y}{2}, \quad \xi \to \operatorname{arccosh}\frac{e^{t/2}}{\sqrt{2}} \tag{4.2}$$

which cancels the $1 - \lambda^2$ factor in the dz^2 term, while maintains a $\sqrt{1-\lambda^2}$ in the cross term; now putting $\lambda = 1$ results in the diagonal solution (3.4).

The fact that this solution admits a G_4 symmetry group, is not due to a prior assumption of this symmetry, but emerged out of the particular nature of the solution of the Painlevé equation. Of course, as shown in detail in section 3, the same solution can be retrieved by first assuming the existence of the fourth killing field. It is interesting that the form of this field is almost dictated by the unknown existence of a fourth Killing field on the slices: for any line element of the form (3.7), there is a fourth Killing field (3.9). Promoting it to a space-time Killing field we arrive, through the solution of (3.13), to the aforementioned solution.

On the other hand, the solution (2.43), which is of Euclidean signature, admits no extra Killing field and is thus a genuine G_3 geometry. It also contains 2 essential parameters, which implies that the number of essential parameters is not associated to the dimension of the isometry group.

The number of essential constants contained in the solution described by (2.15) is 3, as expected for the Type III vacuum Cosmology: two implicit in the Painlevé transcendent plus one multiplicative constant in front of the line element (2.6), owing to the fact that this line element admits no homothetic Killing field.

We believe that the present work constitutes an adequate explanation for the scattered occurrence of various Painlevé transcendents in the literature on Bianchi solutions and we expect to be able to present the corresponding general solutions, along with the known ones, for all lower Bianchi Types (I-VII), in a forthcoming publication.

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Appendix

As we already mention the equation (3.13) is an Abel equation of the first kind for the function $w(t) := u_1'(t)$. Since there is no general method to obtain the solution of such an equation we have tried a different approach, based on the application of Prelle-Singer algorithm [18].

In brief the Prelle-Singer algorithm (actually semi-algorithm) is a method for finding integrating factors for first order differential equations of the form:

$$y' = \frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \tag{A.1}$$

where $M(x,y)$ and $N(x,y)$ are polynomials with coefficients in the field of complex numbers, $\mathbb{C}$. In [18], Prelle and Singer proved that, if an elementary first integral of (A.1) exists, it is possible to find an integrating factor μ for this equation, i.e.

$$\frac{\partial(\mu\, N(x,y))}{\partial\, x} + \frac{\partial(\mu\, M(x,y))}{\partial\, y} = 0 \tag{A.2}$$

For this purpose they defined the operator

$$\mathcal{D} = N\,\frac{\partial}{\partial x} + M\,\frac{\partial}{\partial y} \tag{A.3}$$

and the Darboux polynomials f_i, i.e. irreducible polynomials that obey

$$\mathcal{D} f_i = f_i\, g_i \tag{A.4}$$

for some polynomial g_i. Then, to find the integrating factor μ, one has to choose a degree $N_D = (\text{degree}(x), \text{degree}(y))$ for the polynomials f_i, calculate them and if a relation of the form

$$\sum_{i=1}^{k} n_i\, g_i = -\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) \tag{A.5}$$

is satisfied, for some non-zero rational numbers n_i, then the integrating factor μ, is given by

$$\mu = \prod_{i=1}^{k} f_i^{n_i} \tag{A.6}$$

Unfortunately the method can not define the value of the degree N_D of the polynomials, that's why it is a semi-algorithmic approach. Nevertheless it guarantees, that if an elementary first integral exists, it can be found.

Returning now to equation (3.13), in order to bring it in a form suitable for the Prelle-Singer method, we apply the transformation $w(t) = \frac{2}{3\,(t-1)\,y(t)}$ resulting to

$$y' = \frac{2\,(y-1)\,(6\,t\,y - 4\,t - 3\,y)}{3\,y\,(4\,t^2 - 7\,t + 3)} \tag{A.7}$$

The Darboux polynomials f_i of degree $N_D = (1,1)$ and their corresponding polynomials g_i for the operator

$$\mathcal{D} = 3\,y\,(4\,t^2 - 7\,t + 3)\,\frac{\partial}{\partial t} + 2\,(y-1)\,(6\,t\,y - 4\,t - 3\,y)\,\frac{\partial}{\partial y}$$

are

$$\begin{array}{ll} f_1 = 4\,t - 3 & g_1 = 12\,t\,y - 12 \\ f_2 = t - 1 & g_2 = 12\,t\,y - 9\,y \\ f_3 = 4\,t - 3\,y & g_3 = 12\,t\,y - 6\,y - 6 \\ f_4 = -t\,y + y + t - 1 & g_4 = 24\,t\,y - 15\,y - 8\,t \end{array} \tag{A.8}$$

Inserting this polynomials in (A.5), we can compute the numbers n_i, which read

$$(n_1, n_2, n_3, n_4) = \left(-\frac{1}{2}, \frac{1}{2}, 1, -\frac{5}{2}\right) \tag{A.9}$$

yielding an integrating factor of the form

$$\mu = \frac{(4\,t - 3\,y)\,\sqrt{t-1}}{(t + y - t\,y - 1)^{5/2}\,\sqrt{4\,t-3}} \tag{A.10}$$

At this stage it is a trivial task to integrate (3.13) for $w(t)$ to obtain

$$G(t,w) := \frac{\left(3\,(t-1)\,w - 2\right)^3\,(t-1)^3\,w}{\left(2\,(t-1)^2\,(t-3)\,w^2 - 2\,(t^2 - 4\,t + 3)\,w - 1\right)^2\,(4\,t-3)} = \mathit{constant}$$

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